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# On the rational approximations to *e*

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We have some consequences on the rational approximations to $e$. In (2), P. Bundschuh has proved the following theorem.

Bundschuh's Theorem. *For all integers $p, q$ such that $q>0$,*

$$\left|e-\frac{p}{q}\right|>\frac{\log\log 4q}{18q^2\log 4q}.$$

In (3), C. S. Davis has proved the following definitive theorem for integers $p, q$ such that $|p|, q$ were sufficiently large.

Davis' Theorem. *For any $\varepsilon>0$ there is an infinity of solutions of the inequality*

$$\left|e-\frac{p}{q}\right|<\left(\frac{1}{2}+\varepsilon\right)\frac{\log\log q}{q^2\log q}$$

*in integers $p, q$. Further, there exists a number $q'=q'(\varepsilon)$ such that*

$$\left|e-\frac{p}{q}\right|>\left(\frac{1}{2}-\varepsilon\right)\frac{\log\log q}{q^2\log q}$$

*for all integers $p, q$ with $q\geqq q'$.*

The aim of this note is to prove the following theorem that is generalization of Bundschuh's Theorem.

Theorem. *Let $p, q$ be positive integers.*

*Let $\frac{p_n}{q_n}$ be the $n$-th convergent of $e$. Then*

$$\left|e-\frac{p}{q}\right|>\frac{\log\log q}{\gamma_N q^2\log q}$$

*for all integers $p, q$ such that $q\geqq q_{3N+1}$ $(N\geqq 9)$, where $\gamma_N$ is any constant such that*

$$\gamma_N>(2+5/(N-1/2))\left(1+\frac{\log\log(2(2N+3)/e)}{\log(N+3/2)}\right).$$

Proof. If $p/q$ is not a convergent of $e$, then

$$\left|e-\frac{p}{q}\right|>\frac{1}{2q^2}.$$

Therefore, the theorem is proved in this case.

We must consider the case that $p/q$ is a convergent of $e$.

The continued fraction of $e$ is

$$e=[a_0, a_1, a_2, a_3, \cdots\cdots]$$
$$=[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \cdots\cdots].$$

In other words, $a_0=2$, and for $m\geqq 1$,

$$a_{3m}=a_{3m-2}=1 \text{ and } a_{3m-1}=2m.$$

Case 1. $n=3m$ $(m\geqq N)$

Since $q_{3m+1}=a_{3m+1}q_{3m}+q_{3m-1}=q_{3m}+q_{3m-1}$, we have

$$\left|e-\frac{p_{3m}}{q_{3m}}\right|>\frac{1}{q_{3m}(q_{3m+1}+q_{3m})}=\frac{1}{q_{3m}(2q_{3m}+q_{3m-1})}>\frac{1}{3q^2_{3m}}$$

As we can see that $\log\log x/\log x$ is monotone decreasing for $x\geqq 16$, we have

$$\frac{\log\log q_{3m}}{\log q_{3m}}\leqq\frac{\log\log q_{27}}{\log q_{27}}=0.\,1251198\cdots\cdots<\frac{1}{3},$$

therefore

$$\left|e-\frac{p_{3m}}{q_{3m}}\right|>\frac{\log\log q_{3m}}{q^2_{3m}\log q_{3m}}.$$

Now we have proved the theorem in this case.

Case 2. $n=3m+1$ $(m\geqq N)$

Since $q_{3m+2}=a_{3m+2}q_{3m+1}+q_{3m}=2(m+1)q_{3m+1}+q_{3m}$, we have

$$\left|e-\frac{p_{3m+1}}{q_{3m+1}}\right|>\frac{1}{q_{3m+1}(q_{3m+2}+q_{3m+1})}=\frac{1}{q_{3m+1}((2m+3)q_{3m+1}+q_{3m})}$$
$$>\frac{1}{2(m+2)q^2_{3m+1}}.$$

Now we must estimate $q_{3m+1}$.

Since $q_{3m+1}\geqq 2(2m+1)q_{3m-2}\geqq 2^2(2m+1)(2m-1)q_{3m-5}\geqq\cdots\cdots$
$\geqq 2^m(2m+1)(2m-1)\cdots\cdots 5\cdot 3\cdot 1$, we have

$$\log q_{3m+1}\geqq m\log 2+\sum_{k=1}^{m}\log(2k+1)\geqq m\log 2+\int_1^{m-1}\log(2x+1)\,dx$$
$$=m\log 2+(m-1/2)\log(2m-1)-(m-2)-(3/2)\log 3$$
$$\geqq(m-1/2)((\log 2-1)+\log(2m-1))=(m-1/2)\log(2(2m-1)/e)$$
$$\geqq(m-1/2)\log(m+3/2)$$

Conversely,

$$q_{3m+1} \leqq 2^m (2m+1)(2m-1) \cdots\cdots 5 \cdot 3 \prod_{k=1}^{m} \left(1+\frac{1}{4(2k+1)(2k-1)}\right).$$

Since

$$\prod_{k=1}^{m}\left(1+\frac{1}{4(2k+1)(2k-1)}\right) \leqq \prod_{k=1}^{\infty}\left(1+\frac{1}{12k^2}\right) = \frac{\sinh(\pi/\sqrt{12})}{(\pi/\sqrt{12})}$$
$$=1.1428\cdots\cdots<2.$$

we have

$$q_{3m+1} \leqq 2^{m+1}(2m+1)(2m-1)\cdots\cdots 5\cdot 3.$$

Therefore

$$\log q_{3m+1} \leqq (m+1)\log 2 + \sum_{k=1}^{m} \log(2k+1)$$
$$\leqq (m+1)\log 2 + \int_1^{m+1} \log(2x+1)\, dx$$
$$= (m+1)\log 2 + (m+3/2)\log(2m+3) - m - (3/2)\log 3$$
$$\leqq (m+3/2)((\log 2 - 1) + \log(2m+3))$$
$$= (m+3/2)\log(2(2m+3)/e),$$
$$\log\log q_{3m+1} \leqq \log(m+3/2) + \log\log(2(2m+3)/e).$$

As we can see that $L(x) = \dfrac{\log\log(2(2x+3)/e)}{\log(x+3/2)}$ is monotone decreasing for $x \geqq 9$. we have

$$\log\log q_{3m+1} \leqq (1+L(N))\log(m+3/2).$$

From these consequenes, we find out

$$\frac{\log\log q_{3m+1}}{\log q_{3m+1}} \leqq \frac{1+L(N)}{m-1/2} = (2+5/(m-1/2))(1+L(N))\frac{1}{2(m+2)}$$
$$\leqq (2+5/(N-1/2))\left(1+\frac{\log\log(2(2N+3)/e)}{\log(N+3/2)}\right)\frac{1}{2(m+2)}$$
$$< \frac{\gamma_N}{2(m+2)}.$$

Therefore

$$\left| e - \frac{p_{3m+1}}{q_{3m+1}} \right| > \frac{\log\log q_{3m+1}}{\gamma_N q^2_{3m+1} \log q_{3m+1}}.$$

Now we have proved the theorem in this case.

Case 3. $n=3m+2$ $(m \geqq N)$

We can prove the theorem in this case similarly in the case 1.

This completes the proof of the theorem.

Corollary. *Let $p, q$ be positive integers such that $q \geqq 2$. Then*

$$\left|e-\frac{p}{q}\right|>\frac{\log\log q}{3q^2\log q}.$$

Proof. It is suffice only to consider that $p/q$ is a $(3m+1)$-th convergent of $e$. If $N=32$, then

$$(2+5/(N-1/2))\left(1+\frac{\log\log(2(2N+3)/e)}{\log(N+3/2)}\right)=2.99505\cdots\cdots$$

Hence we can define $\gamma_{32}$ as follows.

$$\gamma_{32}=3$$

From the theorem, for all positive integers $p, q$ such that $q \geqq q_{3m+1}$ $(m \geqq 32)$,

$$\left|e-\frac{p}{q}\right|>\frac{\log\log q}{3\,q^2\log q}$$

We define $\delta_m$ as follows.

$$\left|e-\frac{p_{3m+1}}{q_{3m+1}}\right|>\frac{1}{2(m+2)\,q^2_{3m+1}}=\frac{\log\log q_{3m+1}}{\delta_m q^2_{3m+1}\log q_{3m+1}}$$

i.e. $\delta_m=\dfrac{2(m+2)\log\log q_{3m+1}}{\log q_{3m+1}}$

We show that $\delta_m \leqq 3$ for $m \leqq 31$.

$\delta_1=2.05\cdots\cdots$, $\delta_2=2.72\cdots\cdots$, $\delta_3=2.79\cdots\cdots$, $\delta_4=2.79\cdots\cdots$,
$\delta_5=2.77\cdots\cdots$, $\delta_6=2.75\cdots\cdots$, $\delta_7=2.73\cdots\cdots$, $\delta_8=2.71\cdots\cdots$,
$\delta_9=2.70\cdots\cdots$, $\delta_{10}=2.69\cdots\cdots$, $\delta_{11}=2.67\cdots\cdots$, $\delta_{12}=2.66\cdots\cdots$,
$\delta_{13}=2.65\cdots\cdots$, $\delta_{14}=2.65\cdots\cdots$, $\delta_{15}=2.64\cdots\cdots$, $\delta_{16}=2.64\cdots\cdots$,
$\delta_{17}=2.63\cdots\cdots$, $\delta_{18}=2.63\cdots\cdots$, $\delta_{19}=2.62\cdots\cdots$, $\delta_{20}=2.62\cdots\cdots$,
$\delta_{21}=2.62\cdots\cdots$, $\delta_{22}=2.61\cdots\cdots$, $\delta_{23}=2.61\cdots\cdots$, $\delta_{24}=2.61\cdots\cdots$,
$\delta_{25}=2.60\cdots\cdots$, $\delta_{26}=2.60\cdots\cdots$, $\delta_{27}=2.60\cdots\cdots$, $\delta_{28}=2.60\cdots\cdots$,
$\delta_{29}=2.59\cdots\cdots$, $\delta_{30}=2.59\cdots\cdots$, $\delta_{31}=2.59\cdots\cdots$ :

This completes the proof.

## References


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(3) C. S. Davis, Rational approximations to $e$, J. Austral. Math. Soc. (Series A) **25** (1978), 497–502.