On the Irrationality of $\sum_{k=0}^{\infty}\frac{1}{(4k+1)^2}$

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§ 1. Introduction.

F. Beukers [2] proved the irrationality of $\zeta(2)$ by means of double integrals. This suggests to prove the irrationality of certain infinite series to be possible.

The aim of this note is to prove the following theorem.

THEOREM. *Define* $H=\sum_{k=0}^{\infty} \frac{1}{(4k+1)^2}$ *. Then H is irrational.*

We give an elementary proof of this theorem using the method of F. Beukers.

Note that $H=\frac{1}{2}(\frac{\pi^2}{8}+G)=1.07483307\cdots$, where G denotes Catalan's constant. It is yet undetermined that whether G is irrational or not.

Throughout this note we denote $\lim_{\varepsilon \to +0} \int_{\varepsilon}^{1-\varepsilon}$ by $\int_0^1$, the lowest common multiple of 1, 2, $\cdots$, n by d_n. The value of d_n can be estimate by

$$d_n = \prod_{\substack{\text{prime} \\ p \leq n}} p^{[\log n/\log p]} < \prod_{\substack{\text{prime} \\ p \leq n}} p^{\log n/\log p} = n^{\pi(n)},$$

and the latter number is smaller than 3^n for sufficiently large n.

§ 2. Proof of the Theorem.

LEMMA 1. *Let r and s, be non-negative integers. If* $r > s$, *then*

$$\text{(a)} \quad \int_0^1 \int_0^1 \frac{x^{4r} y^{4s}}{1-(xy)^4}\, dxdy$$

is a rational number whose denominator is a divisor of d_{4r}^2.
If $r = s$, *then*

$$\text{(b)} \quad \int_0^1 \int_0^1 \frac{x^{4r} y^{4r}}{1-(xy)^4}\, dxdy$$

$$= \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} - \left[\frac{1}{1^2} + \frac{1}{5^2} + \cdots + \frac{1}{(4(r-1)+1)^2}\right].$$

Remark. In case $r = 0$, we let the sum $1^{-2} + 5^{-2} + \cdots + (4(r-1)+1)^{-2}$ vanish.

We omit the proof of Lemma 1 since it is very similar to that of Lemma 1 of [2].

LEMMA 2. *Let* $0 < t < 1$, *and let* $i = \sqrt{-1}$. *Then*

$$\text{(c)} \quad \frac{1}{1-t^4} = \frac{1}{4} \sum_{k=0}^{3} \frac{1}{1-i^k t},$$

$$\text{(d)} \quad |1 - i^k t| \geqq 1 - t \quad (k = 0, 1, 2, 3)$$

and

$$\text{(e)} \quad \sum_{k=0}^{3} \frac{1}{(1-i^k t)^m} > 0 \text{ for any positive integer } m.$$

The proof is easy and will be omitted.

Now we give the details of the proof of our result.

Proof of the Theorem. For a positive integer n consider the integral

$$\int_0^1 \int_0^1 \frac{(1-y)^{4n} P_{4n}(x)}{1-(xy)^4}\, dxdy, \tag{1}$$

where $P_{4n}(x)$ is the Legendre-type polynomial given by $(4n)!\, P_{4n}(x) = (d/dx)^{4n} x^{4n} (1-x)^{4n}$. It is clear from Lemma 1 that integral (1) equals $(A_n + B_n H) d_{4n}^{-2}$ for some $A_n \in \mathbf{Z}$ and $B_n \in \mathbf{Z}$. And it is also clear from Lemma 2 that integral (1) equals

$$\frac{1}{4(4n)!} \sum_{k=0}^{3} \int_0^1 \int_0^1 \frac{(1-y)^{4n} (d/dx)^{4n} x^{4n} (1-x)^{4n}}{1-i^k xy}\, dxdy. \tag{2}$$

After a $(4n)$-fold partial integration with respect to x integral (2) changes into

$$\frac{1}{4}\sum_{k=0}^{3}\int_0^1\int_0^1 \frac{y^{4n}(1-y)^{4n}x^{4n}(1-x)^{4n}}{(1-i^k xy)^{4n+1}}\,dxdy. \qquad (3)$$

It is a matter of straightforward computation to show that

$$\frac{y(1-y)x(1-x)}{1-xy} \leqq \left(\frac{\sqrt{5}-1}{2}\right)^5 \text{ for all } 0<x<1 \text{ and } 0<y<1.$$

From Lemma 2, we have the following inequalities

$$\begin{aligned} 0 &< |A_n+B_nH|\,d_{4n}^{-2} \\ &\leqq \frac{1}{4}\sum_{k=0}^{3}\int_0^1\int_0^1 \frac{y^{4n}(1-y)^{4n}x^{4n}(1-x)^{4n}}{|1-i^k xy|^{4n+1}}\,dxdy \\ &\leqq \int_0^1\int_0^1 \left(\frac{y(1-y)\,x(1-x)}{1-xy}\right)^{4n}\frac{1}{1-xy}\,dxdy \\ &\leqq \left(\frac{\sqrt{5}-1}{2}\right)^{20n}\zeta(2), \end{aligned}$$

and hence

$$\begin{aligned} 0 &< |A_n+B_nH| \leqq d_{4n}^2\left(\frac{\sqrt{5}-1}{2}\right)^{20n}\zeta(2) \\ &< 9^{4n}\left(\frac{\sqrt{5}-1}{2}\right)^{20n}\zeta(2) < (5/6)^{4n} \end{aligned}$$

for sufficiently large n. This implies the irrationality of H.

This completes the proof of the theorem.

References

[1] B. C. Berndt, Ramanujan's Notebooks, Part I, Springer-Verlag, New York, 1985.

[2] F. Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, Bull. London Math. Soc. 11 (1979), 268–272.